Recurrence relations: in principle

1. What, in general, is a ***recurrence relation*** for a function or sequence?

2. Linear recurrence relations:
   (a) What does it mean for a recurrence relation to be ***linear***?
   Give some examples of linear and non-linear recurrence relations.
   (b) What is the ***characteristic polynomial*** for a linear recurrence relation?
   (c) Given roughly what happens to the value of the terms in a linear recursion A_n when we increment n, why might we expect exponentials to be involved in the *closed* formulæ for their terms?
   (d) Briefly outline the procedure for finding a closed formula for the terms of a linear recurrence relation using its characteristic polynomial:
      (i) What do the roots of the characteristic polynomial tell us?
      What do we do when the characteristic polynomial has a repeated root?
      (ii) What role is played by *initial values*?

. . . and in practice

3. Find a closed formula for the terms of the sequence given by $A_0 = 1$ and, for $n \geq 0$, $A_{n+1} = (n+1)A_n$.
[Note that this recurrence relation is *not* linear!]

4. Find closed formulæ for the terms of each of the following recursively-defined sequences:
   (a) $B_0 = 5$ and for $n \geq 0$, $B_{n+1} = B_n$.
   (b) $C_0 = 3$ and for $n \geq 0$, $C_{n+1} = 10C_n$.
   (c) $D_0 = 0$, $D_1 = 1$, and for $n \geq 0$, $D_{n+2} = 3D_{n+1} - 2D_n$.
   (d) $E_0 = 1$, $E_1 = 8$, and for $n \geq 0$, $E_{n+2} = 4E_{n+1} - 4E_n$.
   (*e) $F_0 = 0$, $F_1 = 1$, and for $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$.

Please do these for next class!

5. Consider the recurrence relation $T(n) = T\left(\frac{n}{2}\right) + 1$, and suppose that $T(1) = 1$.
   (a) Find $T(1024)$ and, in general, $T(2^k)$.
   (b) If you change variables via $n = 2^k$ (so $k = \log n$), what is the resulting formula for $T(n)$?

6. Consider the recurrence relation $T(n) = 2T\left(\frac{n}{2}\right) + n$, and suppose $T(1) = 0$.
   (a) Show all steps in computing $T(64)$; along the way, you'll have found $T(2^k)$ for $k = 2, 3, \ldots, 6$.
   (b) What is the general formula for $T(2^k)$?
   If you change variables via $n = 2^k$ (so $k = \log n$), what is the resulting formula for $T(n)$?

7. Carefully do the same as in the previous problem, but now for $T(n) = 2T\left(\frac{n}{2}\right) + n^2$.